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Wave groups in uni-directional surface-wave models

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Abstract. Uni-directional wave models are used to study wave groups that appear in wave tanks of hydrodynamic laboratories; characteristic for waves in such tanks is that the wave length is rather small, comparable to the depth of the layer. In second-order theory, the resulting Nonlinear Schrödinger (NLS) equation for the envelope of the wave group contains the dispersion of the group velocity multiplying the linear term and a 'gen-coefficient' that results from mode generation multiplying the nonlinear term. The signs of these coefficients determine whether experimentally relevant wave groups are possible or not. If the dispersion is modelled in such a way that it is correct for all wave lengths for infinitesimal waves, relevant wave groups are obtained consisting of constituent waves with a certain maximal wave length; other models for the dispersion (such as in the KdV-equation) lead to different results.

Keywords: wave groups, Nonlinear Schrödinger equation, short wave dispersion, towing tanks.

1. Introduction

This paper deals with some aspects of the propagation of surface waves. The geometry and dimensions of interest are motivated by water tanks of the hydrodymanic laboratory MARIN, The Netherlands, and the Indonesian Hydrodynamic Laboratory situated at Surabaya for which the towing tank is 235 meters long, 11 meters wide and has an undisturbed water depth of 5·5 m. These tanks are used to test the performance of ships in operational conditions; the waves to be generated should therefore resemble waves of partly or fully developed waves at seas. For an adequate operation of the laboratories, understanding of the basic wave patterns is required. Although the basic physical laws describing the water motion are known, they are rather difficult and require the use of theoretical methods and numerical simulations to predict the wave motion. In this paper some theoretical investigations are presented that deal with the evolution of wave groups; for that aim a KdV-type of equation for uni-directional waves will be used. The results obtained for this approximate model concern the asymptotic description of the evolution of wave groups; necessarily, these results are limited to waves satisfying the underlying assumptions, but the mathematical methods are more transparent than those which use the full set of equations. Besides that, the dependence of the results on the dispersive properties of the model will be investigated. Scaling of the problem, essentially with the depth of the water, makes it clear that wave groups in shallow coastal areas can also be studied with this model. In that respect, the recent numerical calculations on wave groups by De Haas and Zandbergen [1] and De Haas *et al*. [2], De Haas [3] can be mentioned; these results show that small deviations from an initially steady wave group give rather important disturbances on realistic large time and length scales.

A full description of the surface waves would require to solve the fluid equations in the interior. Based on the fact that, for infinitesimally small waves, the effect of the interior fluid motion on the surface can be calculated exactly, and restricting to waves running in one direction, we will use an uni-directional model for the surface elevation. In more detail, the equation for the surface elevation $u = u(x, t)$ is the first-order-in-time equation with dispersion operator R, and the characteristic quadratic nonlinearity:

$$\partial_t u(x,t) = -\partial_x [Ru(x,t) + \tfrac{3}{4} u(x,t)^2]. \tag{1}$$

The variables appearing here are scaled: the spatial variable and the elavation with the depth of the fluid (assumed to be constant), and the time accordingly to scale the velocity of infinitesimal small waves to unity.

Taking for R the simple differential operator $R_{\mathrm{KdV}} := 1 + \partial_x^2/6$, we observe that Equation (1) is the well known Korteweg–de Vries (KdV) equation. The dispersion relation for monochromatic solutions of the form $\mathrm{e}^{i(kx-\omega t)}$ is $\Omega_{\mathrm{KdV}} = k - k^3/6$. Comparing this with the dispersion as found from the full surface wave equations for infinitesimally small waves, which reads

$$\Omega = k\hat{R}(k) \ \text{ with } \ \hat{R}(k) = \sqrt{\frac{\tanh\, k}{k}}, \tag{2}$$

we see that R_{KdV} is the approximation to the first two terms in a Taylor expansion of $\hat{R}$ for small wave numbers k, *i.e.* for sufficiently long waves.

For the applications in hydrodynamic laboratories relatively short wave lengths are considered (somewhat larger than, but of the order of the water depth, correponding to scaled values of k becoming as large as 5, see [4]), while the wave heights never exceed 10 percent of the depth of the basin. This motivates us to describe the dispersion with the operator R, although the method will be generally applicable and also other choices for the dispersion will be investigated.

In Section 2 second-order theory for wave groups with envelopes that are long compared to the wave length of the constituent waves will be considered (for arbitrary operator R). In a frame moving with the group velocity, the amplitude A of the first-order harmonic is a function of ‘slow’ length and time variables ξ, τ, and has to satisfy a Nonlinear Schrödinger (NLS) equation of the form

$$\partial_\tau A + i\beta\partial_\xi^2 A + i\gamma |A|^2 A = 0. \tag{3}$$

The NLS is well-known as amplitude equation for modulated waves in water waves, see *e.g.* Dingemans [5] and the many references therein, and appears in other wave problems as well, see *e.g.* [6, 7].

The sign of the parameters β and γ determine the kind of solutions that are possible; when both are positive, physically relevant (steady) wave groups are possible, for instance exponentially confined as well as modulated periodic wave groups; with opposite signs, the steady solutions are periodic modulations resembling beats and the limiting case of a ‘dark’ soliton with different equilibrium levels at infinity.

In Section 3 it is shown that γ, to be called the ‘gen-coefficient’, results from the nonlinear mode generation by the quadratic nonlinearity; it depends completely on global properties of the dispersion (as is also shown for deep-water and for shallow-water waves, see Dingemans [5]). Investigation of its sign will result in the conclusion that exponentially confined wave groups can exist for the model with dispersion given by the full operator R, provided the wave length of the constituent waves is not too large; this agrees (even quantitatively) with results from the Davey–Stewartson equations. It is also shown that other choices for the dispersion may lead to other results; for instance, dispersion as in the KdV model does not describe confined wave groups.

2. Second-order theory for wave groups

Starting point for the notion of wave groups are monochromatic solutions of *linear* dipersive wave equations. These are solutions of the form

$$a\mathrm{e}^{i\theta} \quad \text{with} \quad \theta = k_0 x - \omega_0 t, \tag{4}$$

where the wave number k_0 and the frequency ω_0 are related by the dispersion relation: $\omega_0 = \Omega(k_0)$. For any amplitude a this is an exact solution. The word 'monochromatic' refers to the fact that only one wave length, *i.e.* one wave number, is present: the spectrum is a Dirac delta function at k_0.

The linearity of the equation implies that a superposition of such monochromatic waves is again a solution. In particular, when a superposition is considered of waves with wave numbers near k_0, so waves with only slightly different wave lengths, the phenomenon of interference appears: reinforcement of the amplitude at some places, and (partial) cancellation at other places. The narrower the spectral function of the superposed waves, the broader in physical space the envelope of the resulting wave group becomes.

For the linear equation the solution, with $\hat{f}$ as initial spatial spectral function, is given by

$$u(x,t) = \int \hat{f}(k)\,\mathrm{e}^{i(kx-\Omega(k)t)}\,\mathrm{d}k.$$

For a Gaussian initial spectral function (centered around $k_0 = 2{\cdot}5$, with standard deviation $0{\cdot}2$), a plot of the initial profile is given in Figure 1 and is characteristic for an (exponentially) confined wave group: for most values of x the cancellation from interference is complete. A superposition of a few monochromatic waves leads to partial cancellation, and therefore to an oscillating profile as shown in Figure 2 for three modes; waves of this last type are characteristic for laboratory experiments.

The profiles of both wave groups show the spatial periodic oscillations with the central wave number k_0 and the envelope with a modulation as a consequence of interference. Observe that the spatial extension L of the envelope is much larger (for the narrow band of wave numbers taken here) than the wavelength of the carrier wave (which is $2\pi/k_0$).

The dynamic behaviour of the wave groups as predicted by linear theory should be an undisturbed translation of the envelope with a speed equal to the group velocity. This can be illustrated most clearly for the confined wave group as is done in Figure 3 for which the values of wave number, frequency, group and phase velocity are given by $k_0 = 2{\cdot}5$, $\omega = 1{\cdot}57$, $V_{\text{group}} = 0{\cdot}335$, $V_{\text{phase}} = 0{\cdot}628$. For illustration the profiles are shifted vertically upwards with increasing time. Observe the difference in phase velocity (the velocity of each wave) and the group velocity with which the profile as a whole is translated.

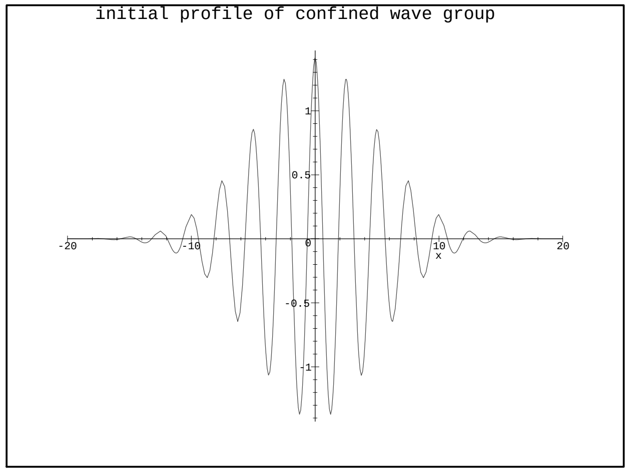

Figure 1. Profile of an exponentially confined wave group.

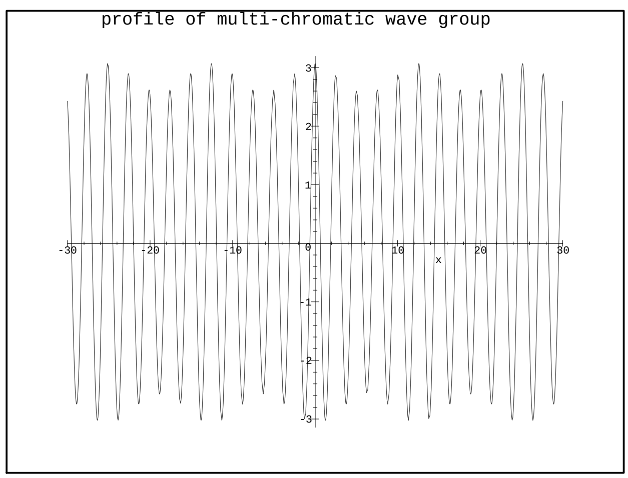

Figure 2. Profile of an oscillating wave group.

The spatial extension and the evolution of the characteristic wave pattern motivates to describe the pattern as an amplitude $a(x, t)$, the envelope, times the periodic carrier wave $\mathrm{e}^{i\theta}$,

$$w(x, t) = a(x, t)\,\mathrm{e}^{i\theta}, \tag{5}$$

and to investigate the equation to be satisfied by a in order that w will be an exact solution of the linear equation. In doing so, and looking for envelopes of large spatial extension, we are led to introduce a scaling in the spatial variable: if ε is the inverse of a measure for the length of the envelope, a long spatial variable ξ is introduced in a frame moving with velocity V (yet unspecified):

$$\xi := \varepsilon(x - Vt) \quad \text{and} \quad b(\xi, t) := a(x, t).$$

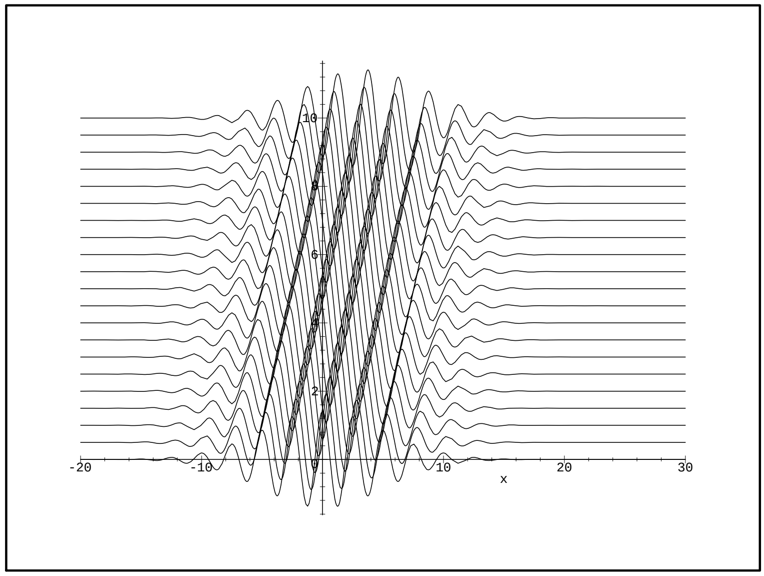

Figure 3. Time evolution of the confined wave group.

Writing the linear equation with dispersion relation Ω in operator form like

$$\partial_t u = -i\Omega(-i\partial_x)u, \tag{6}$$

we observe that the action of the spatial derivative on the product $w = b(\xi, \tau)\,\mathrm{e}^{i\theta}$ can be written like

$$\partial_x w = [ik_0 b(\xi, t) + \varepsilon\partial_\xi b(\xi, t)]\,\mathrm{e}^{i\theta}$$

and that a formal Taylor expansion of Ω leads to the equation to be satisfied by $b(\xi, t)$

$$-i\omega_0 b + \partial_t b - \varepsilon V\partial_\xi b + i\sum_{m=0}\frac{(-i\varepsilon)^m}{m!}\Omega^{(m)}(k_0)\partial_\xi^m b = 0,$$

where $\Omega^{(m)}$ denotes the mth order derivative. Observing that $\omega_0 = \Omega(k_0)$, and taking for V the group velocity, we find in lowest order the equation

$$\partial_t b + i\varepsilon^2\beta\partial_\xi^2 b - \frac{1}{3!}\varepsilon^3\tfrac{1}{3}\,\Omega^{(3)}(k_0)\partial_\xi^3 b = \mathcal{O}(\varepsilon^4),$$

where

$$\beta = -\tfrac{1}{2}\Omega^{(2)}(k_0)$$

is the so called *coefficient of dispersion of the group velocity*. When it is nonzero, the changes in b are over a time scale of order $\mathcal{O}(\varepsilon^2)$; for smaller times this shows that a wave group is propagating with the group velocity without considerable distortion.

For asymptotic results on larger time scales, we also introduce a slow time variable $\tau = \varepsilon^2 t$; collecting the transformations and defining the amplitude function A

$$\xi = \varepsilon(x - \Omega^{(1)}(k_0)t), \qquad \tau = \varepsilon^2 t, \qquad A(\xi, \tau) = a(x, t), \tag{7}$$

we arrive at the linear amplitude equation in lowest order

$$\partial_\tau A(\xi, \tau) + i\beta\partial_\xi^2 A(\xi, \tau) = \mathcal{O}(\varepsilon).$$

REMARK. Another interpretation of the resulting amplitude equation is found when we consider the dispersion relation for this (dispersive) amplitude equation: for modes of the form $\hat{A}(\kappa)\, \mathrm{e}^{i(\kappa\xi - \nu\tau)}$, the dispersion relation reads

$$\nu = -\beta\kappa^2.$$

Multiplied by the carrier wave, there results in the orginal variables

$$\hat{A}(\kappa) \exp[i\{(k_0 + \varepsilon\kappa)x - (\Omega(k_0) + \varepsilon V_{\mathrm{group}}(k_0)\kappa - \varepsilon^2\beta\kappa^2)t\}]$$

which is an expansion in the frequency of the monochromatic wave with wave number $k_0 + \varepsilon\kappa$ of the original linear equation

$$\hat{A}(\kappa) \exp[i\{(k_0 + \varepsilon\kappa)x - \Omega(k_0 + \varepsilon\kappa)t\}].$$

For given ε small and $\hat{A}$ of bounded support, a superposition corresponds to a spectral function in the original variables that is narrowly confined near the value k_0.

The effects of nonlinearity on the shape and propagation of wave groups will now be investigated. No exact theory is possible, so for theoretical investigations we have to be satisfied with approximations. For the linear equations considered above, it is not necessary to restrict to real solutions; for nonlinear equations we have to make this restriction from the onset. Also, mode generation, not present in linear equations, will occur since the nonlinear term in the equation $\partial_x(u^2)$ causes the generation of wave modes that may originally not have been present in the spectrum; a combination of modes $\mathrm{e}^{i(k_1x - \omega_1 t)}$ and $\mathrm{e}^{i(k_2x - \omega_2 t)}$ will lead to modes with wave numbers $k_1 \pm k_2$, and frequencies $\omega_1 \pm \omega_2$. Observe that this mode generation is a typical nonlinear phenomenon; for the linear equation only modes present at the initial time will be present at later times.

However, the nonlinearity also multiplies the amplitudes; for that reason intially small modes will generate other modes of even smaller amplitude. By restricting the interest to first- and second-order effects, we are motivated to look for wave groups as a superposition of the first-order harmonic, a second-order double harmonic mode and a second-order non-harmonic long wave:

$$w(x, t) = \varepsilon\, A(\xi, \tau)\, \mathrm{e}^{i\theta} + \varepsilon^2[B(\xi, \tau) + C(\xi, \tau)\, \mathrm{e}^{2i\theta}\,] + c.c., \tag{8}$$

where A, B and C are functions of the slow variables, and $c.c$ means complex conjugates of all foregoing terms. Substituting this Ansatz in the nonlinear equation

$$\partial_t u = -i\Omega(-i\partial_x)u - \tfrac{3}{4}\,\partial_x u^2, \tag{9}$$

we find that the residue is of the following form

$$\text{residue} = \sum_{n,m} \varepsilon^n R_{nm}\, \mathrm{e}^{im\theta} + c.c.$$

with $n \geqslant 1, m \geqslant 0$; the coefficients R_{nm} contain expressions in A, B, C and their derivatives.

Satisfaction of the dispersion relation makes all terms R_{nm} with $n = 1$ to vanish. For $n = 2$ the term R_{21} vanishes if we take for V the group velocity as in the linear case. The only interesting term is R_{22} which includes the result of the generation of the double harmonic. Requiring it to vanish, we specify the function C as the following expression in A

$$C(\xi, \tau) = \frac{27}{2} \frac{k_0 A(\xi, \tau)^2}{2\Omega(k_0) - \Omega(2k_0)}.$$

For $n = 3$ there are two terms that are of interest. R_{30} reads

$$R_{30} = \frac{27}{4} \partial_\xi |A(\xi, \tau)|^2 - [\Omega^{(1)}(k_0) - \Omega^{(1)}(0)] \partial_\xi B(\xi, \tau)$$

and vanishes if the function B is expressed in terms of A and some ‘constant’ of integration α (possibly depending on τ)

$$B(\xi, \tau) = \frac{27}{4} \frac{|A(\xi, \tau)|^2}{\Omega^{(1)}(k_0) - \Omega^{(1)}(0)} + \alpha(\tau).$$

Finally, the term R_{31} has to vanish to prevent ‘resonance’, *i.e.* as a solvability condition for asymptotically valid solutions. Substituting the expressions for C and B found above, we observe that there results a dynamic equation for A, the amplitude equation

$$\partial_\tau A + \tfrac{1}{3} i k_0 \alpha A + i \beta \partial_\xi^2 A + i \gamma |A|^2 A = 0.$$

The appearance of the constant of integration $\alpha(\tau)$ can easily be ‘gauged’ away by a simple time-harmonic multiplication with $\exp[-i \frac{1}{3} k_0 \int \alpha(\tau)]$. In doing so, we find that the resulting equation is of the form of the Nonlinear Schrödinger Equation (3), where the coefficients are given by

$$\beta = -\tfrac{1}{2} \Omega^{(2)}(k_0), \qquad \gamma = \tfrac{9}{4} k_0 \{\sigma_0 + \sigma_2\},$$

with

$$\sigma_0 = \frac{1}{\Omega^{(1)}(k_0) - \Omega^{(1)}(0)} \quad \text{and} \sigma_2 = \frac{k_0}{2\Omega(k_0) - \Omega(2k_0)}.$$

The coefficient β is the same as for linear wave groups and depends only on the value of k_0.

REMARK. An alternative derivation of the amplitude equation is possible if we exploit the Hamiltonian structure of the original Equation (1). The procedure resembles Whitham averaging technique, see [8], and its generalizations described in [9]; briefly the procedure is as follows.

The Hamiltonian structure of (1) reads

$$\partial_t u = -\partial_x \, \delta\mathcal{H}(u), \tag{10}$$

where $\delta\mathcal{H}(u)$ is the variational derivative of the Hamiltonian $\mathcal{H}(u)$ which is given by

$$\mathcal{H}(u) = \int [\tfrac{1}{2}uRu + \tfrac{1}{4}u^3]\,\mathrm{d}x.$$

Restricting the Hamiltonian by inserting the Ansatz of a wave group and expanding the structure map ∂_x, we find a set of equations from which B and C can be solved with the same result as above. On this subset, the resulting equation is the Hamiltonian formulation of the final NLS equation:

$$\partial_\tau A = i\,\delta\mathcal{K}(A)$$

with Hamiltonian

$$\mathcal{K}(A) = \int [\tfrac{1}{2}\beta|\partial_\xi A|^2 - \tfrac{1}{4}\gamma|A|^4]\,\mathrm{d}\xi.$$

3. Characteristic wave groups

Having found the governing NLS equation for the amplitude, we shall investigate in this section some characteristic properties of its solutions. The properties will depend on the sign of the coefficients β and γ, so this will be investigated first; it turns out that these signs depend on the dispersive properties.

The two contributing terms in the coefficient γ have a clear physical interpretation. The term σ_2 results from the generation of the second harmonic by the nonlinearity; it depends on the difference of the frequency at the original wave number k_0 and at the double harmonic $2k_0$. This difference should be nonzero in order to be able to surpress the resonance, and is generally called a *non-resonance,* or *solvability condition.* If the dispersion is normal in the interval of wave numbers $[k_0, 2k_0]$, this term σ_2 is positive.

Likewise, the term σ_0 results from the generation of the non-harmonic wave by the nonlinearity, and σ_0 depends on the difference of the group velocity at the original wave number k_0 with that for long waves. If the dispersion is normal in the interval $[0, k_0]$, this term σ_0 is negative.

Having shown that γ is completely determined by effects from the generation of modes by the nonlinearity, we will call the *gen-coefficient*. Observe that even in regions of normal dispersion, the signs of the two terms σ_0 and σ_2 are opposite, and so no easy conclusion about the sign of γ can be drawn. As we will see below, the signs of β and γ determine the kind of wave groups that are described by the model.

An investigation of γ as a function of k shows that for the dispersion described by the operator R of infinitesimal surface waves, $\gamma(k)$ is positive only for $k \geqslant k_{\text{crit}}$, where $k_{\text{crit}} \approx 1{\cdot}15$; see Figure 6. This value is close to the critical value $\approx 1{\cdot}362$ that is found when wave groups are considered for the full set of equations describing surface waves on a layer of fluid; the expression obtained here for γ is essentially simpler, see *e.g.* Dingemans [5].

The solutions of the NLS Equation (3) determine the evolution of the envelope of the wave group. If we write the complex amplitude function A with its modulus and phase like

$$A(\xi,\tau) = a(\xi,\tau)\,\mathrm{e}^{i\varphi(\xi,\tau)},$$

the resulting expression for the wave group reads

$$w(x,t) = 2\varepsilon a(\xi,\tau)\,\cos(\psi) + \tfrac{1}{6}\varepsilon^2 a(\xi,\tau)^2\{\sigma_0 + 2\sigma_2\,\cos(2\psi)\}, \tag{11}$$

where the combined phase function is given by

$$\psi(x,t) = k_0 x - \omega_0 t + \varphi(\xi,\tau).$$

Solutions of NLS depend very much on the sign of the coefficients β and γ: different classes of solutions can be found. We may demonstrate this most clearly by looking for 'steady' solutions of the form

$$A(\xi,\tau) = qf(q\xi)\,\mathrm{e}^{-\pm iq^2\tau}.$$

The arbitrary parameter q, from symmetry in the NLS, can and will be put equal to one, since in the expression for the wave groups it can be incorporated in ε. The resulting expression for the wave group then reads

$$w(x,t) = 2\varepsilon f(\xi)\cos(\psi) + \tfrac{1}{6}\varepsilon^2 f(\xi)^2\{\sigma_0 + 2\sigma_2\cos(2\psi)\} \tag{12}$$

with $\psi = k_0 x - (\omega_0 \pm \varepsilon^2)t$.

The function f has to satisfy a second-order ordinary differential equation, given by

$$\beta f'' = \pm f - \gamma f^3.$$

The characteristic solutions of this equation depend essentially on the sign of the coefficients β and γ. Only when both coefficients are positive exponentially confined wave groups, and wave groups with periodically oscillating envelope as shown in Figures 1 and 2 are possible. We can see this by interpreting the equation for f as a mechanical system: Newton's equation of motion for a particle with mass β under the influence of a conservative force with potential energy

$$V(f) = -(\pm\tfrac{1}{2}f^2 - \tfrac{1}{4}\gamma f^4).$$

For positive value of γ and the plus sign, the graph of this function to be called $V_+(f)$ is shown in Figure 4; this shows the existence of a homoclinic orbit (the exponentially confined wave group) and periodic oscillations around $f = \pm 1/\sqrt{\gamma}$.

For negative γ, and the minus sign, the graph of $V_-(f)$ is also shown; now a heteroclinic orbit and periodic oscillations around zero exist.

For $\gamma > 0$, the envelope of the exponentially confined wave group can be written down explicitly, and is given by the well-known formula for the 'bright soliton'

$$f(\xi) = \sqrt{1/\gamma}\,\mathrm{sech}(\xi/\sqrt{\beta}).$$

The periodic solutions can be expressed in terms of elliptic functions; the explicit expressions will not be given here (see *e.g.* [5, 6]). For negative γ, the heteroclinic orbit, with tanh-profile, is the dark soliton; the periodic solutions are 'nonlinear beats'.

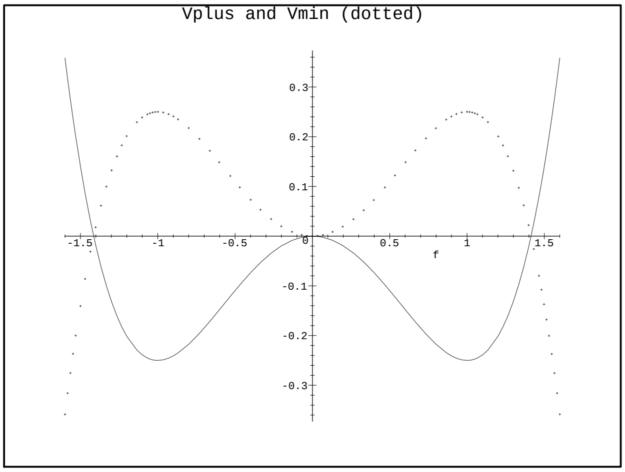

Figure 4. Graphs of the functions V_+ and V_- (dotted) for $\gamma = 1$.

For normal dispersion the signs of the coefficients σ_0 and σ_2 are the same for all wave numbers; these signs are opposite and such that, for the exact dispersion relation, their combination in γ changes the sign of γ from negative for $k \leqslant k_{\text{crit}}$ to positive for $k \geqslant k_{\text{crit}}$. The effect of each contribution in the final expression (11) for the wave group is easily noticed. Since the sign of σ_2 is positive, the effect of the contribution $\varepsilon^2 a(\xi, \tau)^2 \sigma_2 \cos(2\psi)$ in the expression (11) for the wave group is a 'steepening' of the wave, as shown in Figure 5, leading to the characteristic profiles of nonlinear waves, steep crests and flatter valleys, as in the 'cnoidal waves' of the KdV-equation. The fact that σ_0 is negative reveals itself in a depression of the equilibrium surface level over the length of the wave group.

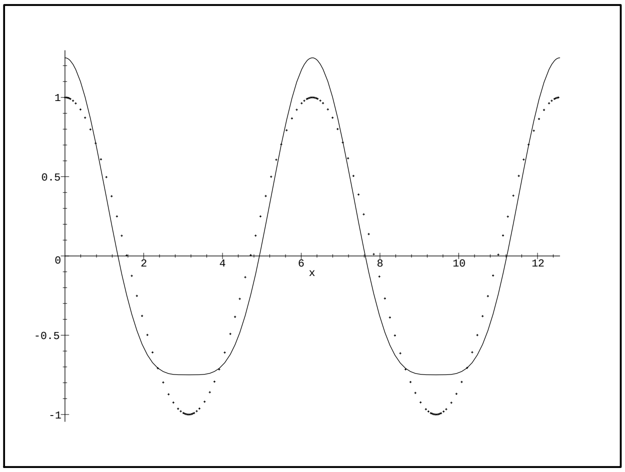

Figure 5. Second harmonic added to the first harmonic (dotted).

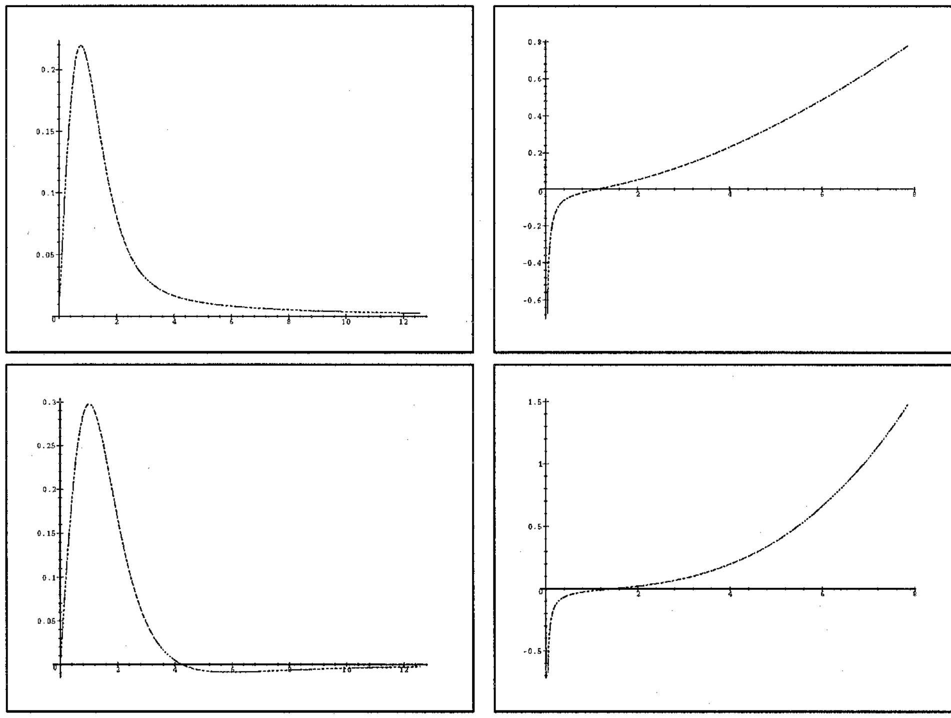

Figure 6. Parameters β (left) and γ for full (top) and BBM – dispersion.

REMARK. For wave groups composed of long waves, *i.e.* $k_0 \mapsto 0$, a Taylor expansion of the expression for the two parameters leads to

$$\beta = \tfrac{1}{2}k_0, \qquad \gamma = -\tfrac{9}{4}k_0^{-1};$$

the same result is obtained for all wave numbers when dispersion is described as in the KdV equation. Hence, in these cases the sign of β is positive (normal dispersion), but the value of the gen-coefficient is negative now; as a consequence, wave groups with profiles as in Figures 1 and 2 are not described.

Another choice for approximate linear dispersion would be as in the BBM-equation: $\omega = k(1 + \frac{1}{6}k^2)^{-1}$. Then the sign of γ is positive for $k \geqslant\approx 1{\cdot}46$, but the dispersion is only normal for wave numbers $k^2 < 18$. Hence outside the range $1{\cdot}46 < k < 4{\cdot}24$ the signs of β and γ are opposite (see Figure 6).

Since wave groups that are relevant for laboratory experiments only exists provided the signs of β and γ are both positive, the results above can be summarized as in the following table:

model	*dispersion relation*	*laboratory wave groups*
full	$k\sqrt{\tanh k/k}$	for $k \geqslant k_{\text{crit}} \approx 1{\cdot}15$
BBM	$k(1 + \frac{1}{6}k^2)^{-1}$	for $1{\cdot}46 \leqslant k \leqslant \sqrt{18} \approx 4{\cdot}24$
KdV	$k(1 - \frac{1}{6}k^2)$	impossible

4. Conclusions

For uni-directional wave equations with various choices for the dispersion, it was shown above how the amplitude equation for wave groups depend on the dispersive properties of the model. The model with dispersion for infinitesimal waves of arbitrary wave length can describe exponentially confined and oscillating wave groups with carrier waves for which the normalized wave lengths do not exceed the value $2\pi/1{\cdot}15 \approx 5{\cdot}46$; for waves on a layer of fluid of depth H this restricts the wave length to $5{\cdot}46H$. For the specific case studied by De Haas and Zandbergen ([1, 2, 3]), the depth is $H = 12$ m and exponentially confined and oscillating wave groups are possible for the wave length of 50 m that is considered in these papers.

For experiments in hydraulic laboratories, the waves of interest have normalized wave length of at most 4 (see [4]); hence, exponentially confined and oscillating wave groups are also possible there. In these experiments the most relevant wave groups are those with slowly varying amplitude that are determined by the displaced – periodic solutions of the NLS equation of which a typical profile is shown in Figure 2.

In a subsequent publication [10] a nonlinear theory for finite-amplitude wave groups will be investigated, since this leads to some modifications of the second-order theory that are relevant on the time and length scales in the laboratory situation.

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References

1. P. C. A. de Haas and P. J. Zandbergen, The application of domain decomposition to time-domain computations of nonlinear water waves with a panel method. *J. Comp. Phys.* 129 (1996) 332–344.
2. P. C. A. de Haas, M. W. Dingemans, and G. Klopman, Simulation of propagating nonlinear wave groups. In: *Proc. 25th Int. Conf. on Coastal Engineering*, Orlando (1996) 165–174.
3. P. C. A. de Haas, *Numerical Simulation of Nonlinear Water Waves in 2-D and 3-D using a Panel Method*; *Domain Decomposition and Applications*. Thesis, University of Twente (1997) 191 pp.
4. E. van Groesen, F. S. Widoyono and T. Nusantara, Modelling waves in a towing tank. *J. Indonesian Math. Soc.* (1998) to appear.
5. M. W. Dingemans, *Water Wave Propagation over Uneven Bottoms*, Part 2. Singapore: World Scientific (1997).
6. N. N. Akhmediev and A. Ankiewicz, *Solitons, Nonlinear Pulses and Beams*. London: Chapman & Hall (1997) 335 pp.
7. C. C. Mei, *Mathematical Analysis in Engineering*. Cambridge: Cambridge University Press (1995) 461 pp.
8. G. B. Whitham, *Linear and Nonlinear Waves*. New York: J. Wiley (1974) 636 pp.
9. E. van Groesen and E. M. de Jager, *Mathematical Structures in Continuous Dynamical Systems*. Amsterdam: Elsevier (1994) 617 pp.
10. E. van Groesen and J. H. Westhuis, Finite amplitude wave groups for hydrodynamic laboratories (1998). In preparation.